The inverse map of a continuous bijective map might not be continuous

The following is a well known fact, whose proof is already covered in class (as an in-class exercise).

Fact: Let X and Y be two topological spaces. Assume X is compact and assume that Y is Hausdorff. Let $f\colon X \to Y$ be a continuous map such that f is also bijective. Then f^{-1} is a continuous map from Y to X.

If X is not assumed to be compact, then for a bijective map $f\colon X \to Y$, f being continuous cannot ensure that f^{-1} is also continuous.

One typical example is like this: Let $X = [0,1)$ and let $Y = S^1 \subset \mathbb{C}$. Define $f\colon X \to Y,\ x \mapsto e^{2\pi i x}$. One can immediately check that f is continuous, f is bijective, but f^{-1} is not continuous. To be more specific, f^{-1} is not continuous at one point $(1,0) \in S^1$, and continous anywhere else (other than the one single point $(1,0)$).

In the example above, there point $0 \in X$ is an end point of X. It is not hard to see that $[0,1)$ above is not a one-dimensional topological manifold (by "manifold" in this note, we mean the manifold in its original definition. Those manifolds with non-trivial boundaries, such as $[0,1)$, are not considered as "manifolds" in this note). In fact, we have the following proposition.

Prop. Let X and Y be two one-dimensional topological manifolds, and let $f\colon X \to Y$ be a continuous map which is bijective. Then f^{-1} is also continuous. *(Note that we do *not* require X to be compact.)*

Idea of proof: Note that continuity is a local property. In order to check/prove that f^{-1} is continuous, we just need to check that f^{-1} is continuous at each point. For any $y \in Y$, as f is bijective, we can find $x \in X$ which is the only element in the preimage of $\{y\}$. Under local charts and by abuse of notations, without loss of generality, we can assume that $x \in (a,b)$ and $y \in (c,d)$. As f is continuous and f is bijective, it follows that (why?)

$$f\left(\left[\frac{a+x}{2}, \frac{b+x}{2}\right]\right) = \left[f\left(\frac{a+x}{2}), f(\frac{b+x}{2}\right)\right].$$

As $\left[\frac{a+x}{2}, \frac{b+x}{2}\right]$ is compact, f is continuous and $f\Big|_{\left[\frac{a+x}{2}, \frac{b+x}{2}\right]}$ is bijective, it follows that $f^{-1}\Big|_{\left[f\left(\frac{a+x}{2}), f(\frac{b+x}{2}\right)\right]}$ is also continuous. Note that $y = f(x) \in \left[f\left(\frac{a+x}{2}), f(\frac{b+x}{2}\right)\right]$, we can claim that f is continuous at y.

Question: Is it possible to find toplogical spaces X and Y, such that X is one dimensional topological manifold (thus having no end-points), f is continuous and bijective, but f^{-1} is not continuous?

Answer: Yes. A typical example is the "lines on torus with irrational slope". Detailed construction is given below.

We identify $\mathbb{R}/\mathbb{Z}$ with $\mathbb{T}$. In $\mathbb{R}/\mathbb{Z}$, note that $[1.4] = [0.4] = [-3.6]$, etc.

Choose an irrational number λ. Let $X = \mathbb{R}$ with the "usual topology" and let Y be $\mathbb{T}^2$ with the usual topology. Define

$$f\colon \mathbb{R} \to \mathbb{T}^2,\ t \mapsto ([t], [\lambda t]).$$

As λ is irrational, it is a well-known fact (and also an interesting and not-so-trivial exercise) that $f(\mathbb{R})$ is dense in $\mathbb{T}^2$. As λ is irrational, it follows immediately that f is injective.

Consider

$$g\colon \mathbb{R} \to f(\mathbb{R}),\ t \mapsto f(t).$$

It is then easy to see that g is both injective and surjective. Let $f(\mathbb{R})$ be endowed with the restricted topology (from $\mathbb{T}^2$ to $f(\mathbb{R})$). As f is continuous, g is also continuous.

Claim: The map g above is continuous, bijective. But g^{-1} is not continuous.

To prove the claim, just use the fact that $f(\mathbb{R})$ is dense in $\mathbb{T}^2$. For $([0],[0]) \in f(\mathbb{R})$, there exists a sequence $\{t_n\}$ such that $[t_n] = [0]$ for all n, $t_i \neq t_j$ if $i \neq j$ and $[\lambda t_n] \to [0]$ as $n \to \infty$. That is, $([t_n],[\lambda t_n]) \to ([0],[0])$. On the other side, as $[t_n] = [0]$ for all n, we have $t_n \in \mathbb{Z}$ for all n. Note that $t_i \neq t_j$ if $i \neq j$. Combined with $t_n \in \mathbb{Z}$ for all n, we have $|t_m - t_n| \geq 1$ for all $m \neq n$. Thus we can claim that t_n does not converge to 0 in $\mathbb{R}$.

So far, we have proved that g^{-1} is not continuous at one point $([0],[0])$. Similarly, we can show that g^{-1} is nowhere continuous. In contrast, the first example we give, $[0,1) \to S^1,\ x \mapsto e^{2\pi i x}$ has only one point of discontinuity.

Remark: In the example above, the domain of g is $\mathbb{R}$, which has no end-point and is a one-dimensional topological manifold. The fact that g^{-1} is not continuous does not conflict with the proposition above. That is because the codomain of g, $f(\mathbb{R})$, is not a one-dimensional topological manifold (why?), although the domain of g is a one-dimensional topological manifold. In fact, if one checks the covering dimension (a.k.a. Lebesgue covering dimension) of $f(\mathbb{R})$, it turns out that the covering dimension of $f(\mathbb{R})$ is two instead of one (this fact should not be quite a surprise though. Just think about the Peano curve). Besides, with covering dimension two, $f(\mathbb{R})$ is not a two dimensional manifold either.

Remark: The example above is about irrational flows on $\mathbb{T}^n$, which also serve as basic/typical examples for noncommutative geometry.